Time Series Analysis

Time Series

- A <u>time series</u> is a sequence of data points, measured typically at successive times spaced at uniform time intervals, i.e., daily exchange rates, weekly stock returns, monthly interest rates and annual GDP.
- We denote a time series as X_t for t = 1, 2, …, n.
- The notation is very different that cross section data, i.e., X_i.
- X_t contains two important elements: a) the time of reference t and b) the magnitude of the series at time t.
- The hierarchy of the sequence of the data is also very important.

ARIMA Models

- In regression analysis the objective is to explain the behaviour of one variable as a function of several other variables, i.e., $Y = f(X_1, X_2, \dots X_k)$.
- In time series analysis this objective does not exit.
- In time series analysis the behaviour of one variable is explained by its past behaviour.
- Thus, using the Box and Jenkins (1970) methodology the objective is to identify the best fitted model obtained by a class of models known as ARIMA models.
- ARIMA stands for **A**uto**R**egressive **I**ntegrated **M**oving **A**verage Processes.

Stationary Processes

- The time series analysis begins with the definition of a stationarity.
- Stationary process is the process that converges to its long run equilibrium.
- It is also a process that its properties are well defined and therefore can be analysed.
- More formally, a process is said to be *strictly stationary* if its properties are unaffected by a change of time origin and its joint probability distribution at any times $t_1, t_2, .., t_m$ must be the same as the joint probability distribution at times t_{1+k}, $t_{2+k}, \dots, t_{m+k}$, where k is an arbitrary shift in time (k is an integer number).
- A simpler version of the above definition for stationary is given by the concept of weak stationarity.
- A process is said to be *weakly stationary* if it satisfies the following three conditions:
  - The mean of the process is constant through the time: $E[X_t] = \mu$ for all t.
  - The variance is constant through the time: $Var[X_t] = \sigma^2$ for all t.
  - The covariance between any two values of the series depends only on their distance apart in time, not in the absolute location in time: $Cov(X_t, X_{t-k}) = \gamma_k$.

Autocovariances

- The quantities γ_k are called <u>autocovariances</u> for k = …. -1, 0, 1, ….
- These quantities are <u>population autocovariances</u>.
- Notes:
  - 1) $\gamma_0 = Var(X_t) = \sigma^2$ which is the variance of the series.
  - 2) $\gamma_{-k} = \gamma_k$ from stationarity.
- The quantities c_k are called <u>sample autocovariances</u>.

$$c_k = \frac{1}{n}\sum_{t=k+1}^{n}(X_t - \bar{X})(X_{t-k} - \bar{X})$$

- where $\bar{X} = \frac{1}{n}\sum_{t=1}^{n} X_t$

Autocorrelations

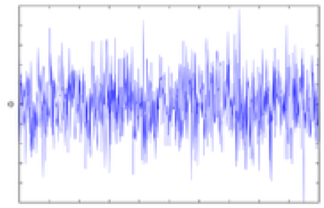
- The <u>population autocorrelations</u> ρ_k are defined as:

$$\rho_k = Corr(X_t,\, X_{t-k}) = \frac{Cov(X_t, X_{t-k})}{\sqrt{Var(X_t)}\sqrt{Var(X_{t-k})}} = \frac{\gamma_k}{\sigma\sigma} = \frac{\gamma_k}{\sigma^2} = \frac{\gamma_k}{\gamma_0}$$

- Note:
  - 1) $\rho_0 = 1$
  - 2) $\rho_{-k} = \rho_k$ from stationarity.
- The graph of a set of autocorrelations is known as <u>correlogram</u>, i.e., graphical presentation of ρ_k for values of k = 0, 1, 2, ….

The white noise process

- The term white noise process comes from the engineering area as a random signal.



- This process has some unique characteristics that have been used in econometrics analysis also, i.e., the error term in any regression model.

The simplest example - W.N.

- As in econometrics, the White Noise (W.N.) process, denoted as ε, has the following characteristics:
  - Mean zero, i.e., $E[\varepsilon] = 0$ for all t.
  - Constant variance, i.e., $Var[\varepsilon] = \sigma^2$ for all t.
  - Uncorrelated values, i.e., $Cov(\varepsilon_t, \varepsilon_s) = 0$ for $t \neq s$.
- Thus, this process has values of ρ_k, such that:

  $\rho_0 = 1$ and all other $\rho_k = 0$ for $k = 1, 2, \ldots$
- Graph the correlogram of this process.

The sample autocorrelations

- The sample autocorrelations r_k are defined as:

$$r_k = \frac{c_k}{c_0} = \frac{\sum_{t=k+1}^{n}(X_t - \bar{X})(X_{t-k} - \bar{X})}{\sum_{t=1}^{n}(X_t - \bar{X})^2}$$

- For moderately large samples, the sample autocorrelations are approximately normally distributed.
- The r_k's of a white noise process have mean zero and variance:

$$Var(r_k) \approx \frac{n-k}{n(n+2)} \quad or \quad Var(r_k) \approx \frac{1}{n}$$

- when k is small relative to n.

The objective

- The aim in time series analysis is to explain what will happen in the future in terms of what is already known.
- In this sense, the mean and the variance of a time series are of limited value.
- The most relevant information is likely to come from autocovariances γ_k and furthermore from the autocorrelations ρ_k.
- So the objective in time series analysis can be summarized as follows:

Data → Model (how to obtain)

↓ ↓

r_k → ρ_k (estimates)

The AR(1) Process

- A stationary time series with mean zero, i.e., $E(X_t)=0$, is generated by an AR(1) process as follows:

$$X_t = \varphi X_{t-1} + \varepsilon_t$$

  where $|\varphi| < 1$, $t = 1, 2, \ldots, n$ and ε_t is a white noise process.
- The model presents short run behavior, whereas the long run behavior is obtained by recursively substituting lag values of X.
- Hence,

$$X_t = \varphi^n X_0 + \sum_{i=0}^{n-1} \varphi^i \varepsilon_{t-i} \quad or \quad X_t = \sum_{i=0}^{\infty} \varphi^i \varepsilon_{t-i}$$

- Since, $|\varphi| < 1$ ➔ $\varphi^n \to 0$ as $n \to \infty$.
- Thus the long run behavior of the series is expressed as a function of ε.
- Example Y_F in Macroeconomics.

Autocorrelations of AR(1)

- The autocovariances of an AR(1) process are:
  $\gamma_0 = \sigma^2/(1 - \varphi^2)$ and $\gamma_k = \varphi\gamma_{k-1}$ for $k \geq 1$.
- Note that for the existence of γ_0 ➔ $|\varphi| < 1$.
- The autocorrelations of an AR(1) process are:
  $\rho_0 = 1$, $\rho_1 = \varphi$ and $\rho_k = \varphi\rho_{k-1}$ for $k \geq 2$ or $\rho_k = \varphi^k$ for $k \geq 0$.
- Note that ρ_k go to zero as k increases.
- For $\varphi = 0.8$ ➔ $\rho_1 = 0.8$, $\rho_2 = 0.64$, $\rho_3 = 0.512$, $\rho_4 = 0.4096$ etc.
- For $\varphi = -0.8$ ➔ $\rho_1 = -0.8$, $\rho_2 = 0.64$, $\rho_3 = -0.512$, $\rho_4 = 0.4096$ etc
- Thus, the autocorrelations follow the model.

The AR(1) with mean

- $X_t \sim AR(1)$ with $E(X_t) = \mu \neq 0$ for all t.
- The model: $X_t - \mu = \varphi(X_{t-1} - \mu) + \varepsilon_t$
- Since the autocorrelations are not affected by the mean the ρ_k will behave as previously presented.
- Constant versus Mean.
- Consider the SRM: $X_t = \alpha + \beta X_{t-1} + \varepsilon_t$
- If both models can equally be estimated then $\beta = \varphi$ and $\alpha = (1 - \varphi)\mu$ so $\mu = \alpha/(1 - \varphi)$.

The Backshift operator

- The Backshift operator B is defined as:
  $B^j X_t = X_{t-j}$ where $j \geq 1$. For $j = 0$ ➔ $B^0 X_t = X_t$.
- AR(1): $X_t = \varphi X_{t-1} + \varepsilon_t$ ➔ $(1 - \varphi B)X_t = \varepsilon_t$.
- Thus, $X_t = 1 / (1 - \varphi B)\varepsilon_t = [\, 1 + \varphi B + \varphi^2 B^2 + \ldots \,]\varepsilon_t$.
- So, the AR(1) process can be written as linear process of infinite past values of the error term.
- This is the long run behavior of the AR(1) process.
- To ensure that the root of the $1/B - \varphi = 0$ must be less that one.
- The series is the infinite sum of a geometric process of ε_t.

The AR(2) Process

- A stationary time series with mean zero, i.e., $E(X_t) = 0$, is generated by an AR(2) process as follows:

$$X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \varepsilon_t$$

where $\varphi_1 + \varphi_2 < 1$ or $-\varphi_1 + \varphi_2 < 1$ and $|\varphi_2| < 1$, t = 1, 2, …, n and ε_t is a white noise process.

- The model presents short run behavior, whereas the long run behavior is obtained by recursively substituting lag values of X.
- Alternatively, $(1 - \varphi_1 B - \varphi_2 B^2)X_t = \varepsilon_t$ ➔ $X_t = 1 / (1 - \varphi_1 B - \varphi_2 B^2)\varepsilon_t$.
- Thus, the AR(2) can be written as a linear process with infinite error terms which is the long run behavior of the series.
- To ensure that the roots of the polynomial $1 / B^2 - \varphi_1 1/B - \varphi_2 = 0$ must be less than one. $\{\varphi_1 + \varphi_2 < 1$ or $-\varphi_1 + \varphi_2 < 1$ and $|\varphi_2| < 1\}$.

Autocovariances of AR(2)

- The autocovariances of an AR(2) process are:

$$\gamma_0 = \frac{(1-\varphi_2)}{(1+\varphi_2)[(1-\varphi_2)^2 - \varphi_1^2]}\sigma^2$$

- Since $\gamma_0 > 0$, then $(1 - \varphi_2) > 0$ ➔ $|\varphi_2| < 1$.
- Also, $(1 + \varphi_2)[(1 - \varphi_2)^2 - \varphi_1^2] > 0$. Since $|\varphi_2| < 1$ ➔ $(1 + \varphi_2) > 0$.
- Thus, $[(1 - \varphi_2)^2 - \varphi_1^2] > 0$ ➔ $[(1 - \varphi_2) - \varphi_1]\,[(1 - \varphi_2) + \varphi_1] > 0$.
- So, $1 - \varphi_2 - \varphi_1 > 0$ ➔ $\varphi_1 + \varphi_2 < 1$ and $1 - \varphi_2 + \varphi_1 > 0$ ➔ $-\varphi_1 + \varphi_2 < 1$.
- Moreover, $\gamma_1 = [\varphi_1 / (1 - \varphi_2)]\,\gamma_0$ & $\gamma_2 = [\varphi_1^2 / (1 - \varphi_2) + \varphi_2]\,\gamma_0$.
- All other autocovariances are defined as:

  $\gamma_k = \varphi_1\gamma_{k-1} + \varphi_2\gamma_{k-2}$ for all values of $k \geq 3$.

Autocorrelations of AR(2)

- The autocorrelations of an AR(2) process are:
  - $\rho_1 = \varphi_1/(1 - \varphi_2)$
  - $\rho_2 = [\varphi_1^2 / (1 - \varphi_2) + \varphi_2]$
  - $\rho_k = \varphi_1\rho_{k-1} + \varphi_2\rho_{k-2}$ for all values of $k \geq 3$.
- Hence, for values of $k \geq 3$ the autocorrelations behave as the AR(2) model.
- The first two autocorrelations are used for the initial conditions.
- The autocorrelations decay toward zero.

The AR(2) with mean

- $X_t \sim AR(2)$ with $E(X_t) = \mu \neq 0$ for all t.
- The model: $X_t - \mu = \varphi_1(X_{t-1} - \mu) + \varphi_2(X_{t-2} - \mu) + \varepsilon_t$.
- Since the autocorrelations are not affected by the mean the ρ_k will behave as previously presented.
- Also, $(1 - \varphi_1 B - \varphi_2 B^2)(X_t - \mu) = \varepsilon_t$.
- Constant versus Mean.
- Consider the SRM: $X_t = \alpha + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \varepsilon_t$
- If both models can equally be estimated then : $\beta_1=\varphi_1$, $\beta_2=\varphi_2$ and $\alpha = (1 - \varphi_1 - \varphi_2)\mu$ so $\mu = \alpha/(1 - \varphi_1 - \varphi_2)$.

The AR(p) Process

- A stationary time series with mean zero, i.e., $E(X_t) = 0$, is generated by an AR(p) process as follows:

  $$X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \ldots + \varphi_p X_{t-p} + \varepsilon_t$$

  ε_t is a white noise process.
- The model presents short run behavior, whereas the long run behavior is obtained by recursively substituting lag values of X.
- Alternatively, $(1 - \varphi_1 B - \varphi_2 B^2 - \ldots - \varphi_p B^p)X_t = \varepsilon_t$ ➔

  $$X_t = 1 / (1 - \varphi_1 B - \ldots - \varphi_p B^p)\varepsilon_t.$$
- Thus, the AR(p) can be written as a linear process with infinite error terms which is the long run behavior of the series.
- To ensure that the roots of $1/B^p - \varphi_1 1/B^{p-1} - \ldots \varphi_p = 0$ must be less than one.

Autocovariances of AR(p)

- The autocovariances of an AR(p) process are:

  $$\gamma_0 = \frac{1}{1 - \varphi_1 \rho_1 - \varphi_2 \rho_2 \ldots - \varphi_p \rho_p} \sigma^2$$
- Then, up to the order of the AR process, that is the first p autocovariances are used for the initial conditions.
- After the order of the process all other autocovariances are defined as:

  $\gamma_k = \varphi_1 \gamma_{k-1} + \varphi_2 \gamma_{k-2} + \ldots + \varphi_p \gamma_{k-p}$ for all values of $k \geq p+1$.

Autocorrelations of AR(p)

- To obtain the first p autocorrelations we use the Yule-Walker equations:
  - $\rho_1 = \varphi_1 + \varphi_2\rho_2 + \varphi_3\rho_3 + \ldots + \varphi_p\rho_{p-1}$
  - $\rho_2 = \varphi_1\rho_1 + \varphi_2 + \varphi_3\rho_2 + \ldots + \varphi_p\rho_{p-2}$
  - …………………………………….
  - $\rho_p = \varphi_1\rho_{p-1} + \varphi_2\rho_{p-2} + \varphi_3\rho_{p-3} + \ldots + \varphi_p$
- These equations will give us a system of p equations with p unknown.
- The first p autocorrelations are used for the initial conditions.
- All other autocorrelations follow the AR(p) model, i.e.,
  $\rho_k = \varphi_1\rho_{k-1} + \varphi_2\rho_{k-2} + \ldots \varphi_p\rho_{k-p}$ for all values of $k \geq p+1$.
- The autocorrelations decay toward zero.

The MA(1) Process

- A time series with mean zero, i.e., $E(X_t) = 0$, is generated by a MA(1) process as follows:
  $$X_t = \varepsilon_t - \theta\varepsilon_{t-1}$$
  where $|\theta| < 1$, $t = 1, 2, \ldots, n$ and ε_t is a white noise process.
- The model presents only short run behavior.
- If the mean of the process is not zero, i.e., $E(X_t) = \mu$ then:
  $X_t - \mu = \varepsilon_t - \theta\varepsilon_{t-1}$
- It is interesting to note that only for one period inference can be made.

Autocorrelations of MA(1)

- The autocovariances of a MA(1) process are:
  - $\gamma_0 = (1 + \theta^2)\sigma^2$
  - $\gamma_1 = -\theta\sigma^2$
  - $\gamma_k = 0$ for all values of $k \geq 2$.
- Hence the autocorrelations of a MA(1) process are:
  - $\rho_0 = 1$
  - $\rho_1 = -\theta/(1 + \theta^2)$
  - $\rho_k = 0$ for all values of $k \geq 2$.
- Note that $|\rho_1| \leq 0.5$, since $(1 + \theta)^2 \geq 0$ ➔ $(1 + \theta^2) \geq -2\theta$ ➔ $|\theta|/(1 + \theta^2) \leq 0.5$
- Thus, only the first autocorrelation is not zero and all other are zero.

Invertibility

- <u>Invertibility</u> means that an MA process can be expressed as an AR process with infinite terms.
- So for the MA(1) process we have: $X_t = \varepsilon_t - \theta\varepsilon_{t-1}$ ➔ $X_t = (1 - \theta B)\varepsilon_t$ ➔ $\varepsilon_t = 1/(1 - \theta B)X_t$ ➔ $\varepsilon_t = [1 + \theta B + \theta^2 B^2 + \dots)X_t$
- To invert a MA(1) process to an AR process we much have $|\theta| < 1$.
- In other words, invertibility requires that the root of the polynomial $1/B - \theta = 0$ must be less than one.
- Inveribility also excludes the possibility of having same models with different values of θ.
- Consider: $X_t = \varepsilon_t - \theta\varepsilon_{t-1}$ and $X_t = \varepsilon_t - \theta^{-1}\varepsilon_{t-1}$, i.e., 0.5 and 2.
- Both models are identical since $\rho_1 = -\theta/(1 + \theta^2) = -0.4$ but the first one has smaller variance.
- Inveritibility should not related to stationarity.

The MA(2) Process

- A time series with mean zero, i.e., $E(X_t) = 0$, is generated by a MA(2) process as follows:

$$X_t = \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}$$

  where ε_t is a white noise process.
- For invertibility, i.e., $X_t = (1 - \theta_1 B - \theta_2 B^2)\varepsilon_t \Rightarrow$ $\varepsilon_t = 1/(1 - \theta_1 B - \theta_2 B^2)X_t$, the roots of the polynomial: $1/B^2 - \theta_1 1/B - \theta_2 = 0$ must be less than one which means that $\theta_1 + \theta_2 < 1$, $-\theta_1 + \theta_2 < 1$ and $|\theta_2| < 1$.
- The model presents only short run behavior.
- If the mean of the process is not zero, i.e., $E(X_t) = \mu$ for all t then: $X_t - \mu = \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}$.
- We can make inference only for two periods.

Autocorrelations of MA(2)

- The autocovariances of a MA(2) process are:
  - $\gamma_0 = (1 + \theta_1^2 + \theta_2^2)\sigma^2$
  - $\gamma_1 = \theta_1(\theta_2 - 1)\sigma^2$
  - $\gamma_2 = -\theta_2\sigma^2$
  - $\gamma_k = 0$ for all values of $k \geq 3$.
- Hence the autocorrelations of a MA(1) process are:
  - $\rho_0 = 1$
  - $\rho_1 = [\theta_1(\theta_2 - 1)]/(1 + \theta_1^2 + \theta_2^2)$
  - $\rho_2 = -\theta_2/(1 + \theta_1^2 + \theta_2^2)$
  - $\rho_k = 0$ for all values of $k \geq 3$.
- Thus, only the first two autocorrelations are not zero and all other are zero.

The MA(q) Process

- A time series with mean zero, i.e., $E(X_t) = 0$, is generated by a MA(q) process as follows:

$$X_t = \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} - \ldots - \theta_q\varepsilon_{t-q}$$

  where ε_t is a white noise process.
- For invertibility, i.e., $X_t = (1- \theta_1 B - \ldots - \theta_q B^q)\varepsilon_t$ ➔
  $\varepsilon_t = 1/(1- \theta_1 B - \ldots - \theta_q B^q)X_t$,
  the roots of the polynomial: $1/B^q - \theta_1\, 1/B^{q-1} - \theta_q = 0$
  must be less than one.
- If the mean of the process is not zero, i.e., $E(X_t) = \mu$ then:
  $X_t - \mu = \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} - \ldots - \theta_q\varepsilon_{t-q}$
- Alternatively, $X_t - \mu = (1- \theta_1 B - \ldots - \theta_q B^q)\varepsilon_t$.

Autocorrelations of MA(q)

- The autocovariances of a MA(q) process are:
  - $\gamma_0 = (1 + \theta_1^2 + \ldots + \theta_q^2)\sigma^2$
  - $\gamma_k \neq 0$ for all values of $k < q$.
  - $\gamma_k = -\theta_q\sigma^2$ for $k = q$.
  - $\gamma_k = 0$ for all values of $k \geq q+1$.
- Hence the autocorrelations of a MA(1) process are:
  - $\rho_0 = 1$
  - $\rho_k \neq 0$ for all values of $k < q$.
  - $\rho_k = -\theta_q/(1 + \theta_1^2 + \ldots + \theta_q^2)$ for $k = q$.
  - $\rho_k = 0$ for all values of $k \geq p+1$.
- Thus, only the first q autocorrelations are not zero and all other are zero.

ARMA(p, q) models

- A time series with mean zero, i.e., $E(X_t) = 0$, is generated by an ARMA(p, q) process as follows:

$X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \ldots + \varphi_p X_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \ldots - \theta_q \varepsilon_{t-q}$
where ε_t is a white noise process.

- Equivalent, $(1-\varphi_1 B-\ldots-\varphi_p B^p)X_t = (1-\theta_1 B - \ldots - \theta_q B^q)\varepsilon_t$ ➔ $\varphi(B)X_t = \theta(B)\varepsilon_t$
- Conditions for stationarity and inveribility must be satisfied.
- No common roots.
- If the process has mean then it is written as:
  $\varphi(B)(X_t - \mu) = \theta(B)\varepsilon_t$

Autocorrelations of ARMA(p, q)

- The autocovariances of an ARMA(p, q) process are:
  - For $k \leq q$ ➔ $\gamma_0, \gamma_1, \ldots, \gamma_k$ depend on φ's and θ's.
  - For $k > q$ ➔ $\gamma_k = \varphi_1\gamma_{k-1} + \varphi_2\gamma_{k-2} + \ldots + \varphi_p\gamma_{k-p}$.
- The autocorrelations of an ARMA(p, q) process are:
  - For $k \leq q$ ➔ $\rho_0, \rho_1, \ldots, \rho_k$ depend on φ's and θ's.
  - For $k > q$ ➔ $\rho_k = \varphi_1\rho_{k-1} + \varphi_2\rho_{k-2} + \ldots \varphi_p\rho_{k-p}$.
- Rule: For k > q the autocorrelations of an ARMA(p, q) model behave as an AR(p) model. The moving average parameters play role only to compute the first q autocorrelations.
- Note the difference between the constant and the mean of the series is determined only by the presence of the autoregressive parameters.

The ARMA(1, 1) Process

- A time series with mean zero, i.e., $E(X_t) = 0$, is generated by an ARMA(1, 1) process as follows:
  $$X_t = \varphi X_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}$$
  where $|\varphi| < 1$, $|\theta| < 1$, $t = 1, 2, \ldots, n$ and ε_t is a white noise process.
- Note:
  - If $\varphi = 0$, the process is a MA(1).
  - If $\theta = 0$, the process is an AR(1)
  - If $\varphi = \theta$, the process is a white noise.
- The process is written also as: $(1 - \varphi B)X_t = (1 - \theta B)\varepsilon_t$
  - If it is stationary $X_t = [1/(1 - \varphi B)](1 - \theta B)\varepsilon_t$
  - If it is invertible $\varepsilon_t = [1/(1 - \theta B)](1 - \varphi B)X_t$
- If the mean of the process is not zero, i.e., $E(X_t) = \mu$, then:
  $(1 - \varphi B)(X_t - \mu) = (1 - \theta B)\varepsilon_t$
- By recursively substituting: $X_t = \varepsilon_t + (\varphi - \theta)\Sigma_{j=1}\varphi^{j-1}\varepsilon_{t-j}$.

Autocovariances of ARMA(1, 1)

- The autocovariances of an ARMA(1, 1) process are:
  - $\gamma_0 = [1 - 2\varphi\theta + \theta^2]/(1 - \varphi^2)\sigma^2$
    - If $\theta = 0$, $\gamma_0 = 1/(1 - \varphi^2)\sigma^2$ ➔ AR(1)
    - If $\varphi = 0$, $\gamma_0 = [1 + \theta^2]\sigma^2$ ➔ MA(1)
    - If $\varphi = \theta$, $\gamma_0 = \sigma^2$ ➔ W.N.
  - $\gamma_1 = [(1 - \varphi\theta)(\varphi - \theta)]/(1 - \varphi^2)\sigma^2$
  - $\gamma_k = \varphi\gamma_{k-1}$ for all values of $k \geq 2$.
- The autocovariances behave like the AR(1) process for values of $k \geq 2$.

Autocorrelations of ARMA(1, 1)

- The autocorrelations of an ARMA(1, 1) process are:
  - $\rho_0 = 1$
  - $\rho_1 = [(1 - \varphi\theta)(\varphi - \theta)]/[1 - 2\varphi\theta + \theta^2]$
    - If $\theta = 0$, $\rho_1 = \varphi$ ➔ AR(1)
    - If $\varphi = 0$, $\rho_1 = -\theta/[1 + \theta^2]$ ➔ MA(1)
    - If $\varphi = \theta$, $\rho_1 = 0$ ➔ W.N.
  - $\rho_2 = \varphi\rho_1$
  - $\rho_k = \varphi\rho_{k-1}$ or $\rho_k = \varphi^{k-1}\rho_1$ for all values of $k \geq 3$.
- Note that θ appears only on ρ_1 and all ρ_k for $k \geq 2$ behave like an AR(1). The only difference with an AR(1) is the initial values of ρ_1 which includes the value of θ.

Integrated Processes

- Let X_t be a non-stationary process in levels, i.e., the mean of the process is not constant through time.
- However, this process can be stationary in first difference, i.e., $W_t = X_t - X_{t-1} = (1 - B)X_t$
- Non-stationary processes occur when at least one of the characteristics of stationary process is violated.
- Recall that a stationary process has constant mean and variance and the covariance between any two values of the series depends only on their distance apart in time, not in the absolute location in time.
- Taking differences we hope that the process will become stationary.
- The number of differences characterizes the order of Integration denoted as I.
- In general we write: $W_t = (1 - B)^d X_t$ where d is a non-negative integer number , i.e., d = 0, 1, 2, …
  - if d = 0 ➔ $W_t = X_t$ and the process is stationary I(0).
  - if d = 1 ➔ $W_t = (1 - B)X_t = X_t - X_{t-1} = \Delta X_t$ and the process is I(1) or stationary in first differences.
  - if d = 2 ➔ $Z_t = (1 - B)^2X_t = (1 - B)W_t = W_t - W_{t-1} = \Delta W_t = X_t - 2X_{t-1} + X_{t-2}$ and the process is I(2) or stationary in second differences.
- An ARIMA(p, d, q) model is written as: $\varphi(B)(1 - B)^dX_t = \theta(B)\varepsilon_t$.
- Non-stationary processes do not converge to long run equilibrium and they are explosive.

A special case: Random Walk

- Consider the AR(1) process: $X_t = \varphi X_{t-1} + \varepsilon_t$
  - If $|\varphi| < 1$ the process is stationary.
  - If $\varphi = 1$ the process is non-stationary (unit autoregressive root).
- Suppose $\varphi = 1$ ➔ $X_t = X_{t-1} + \varepsilon_t$ ➔ $X_t - X_{t-1} = \varepsilon_t$ ➔ $\Delta X_t = \varepsilon_t$ ➔ $(1 - B)X_t = \varepsilon_t$.
- The process $X_t - X_{t-1} = \varepsilon_t$ is known as Random Walk process.
- By recursive substitution we get: $X_t = X_0 + \Sigma_{j=1}^{t} \varepsilon_j$.
- Assuming that the initial value is zero, i.e., $X_0 = 0$, then

  $$X_t = \Sigma_{j=1}^{t} \varepsilon_j$$

  is simply the sum of past values of the error term.
- Short run versus long run.

Characteristics of RW

- The mean of the process is zero, i.e., $E(X_t) = 0$.
- The conditional mean is the past value, i.e., $E(X_t|t\text{-}1) = X_{t-1}$.
- For the Short Run the best we can do to predict X_{t+1} is X_t whereas for the Long Run you can do nothing.
- The variance of the process is $Var(X_t) = Var(\Sigma_{j=1}^{t} \varepsilon_j) = t\sigma^2$.
- Hence the variance is not constant, it depends on t and therefore increases as t increases.
- Autocovariances: Suppose $1<t<s$ then $\gamma_{t,s} = Cov(X_t, X_s) = t\sigma^2$.
- Alternatively, $\gamma_{t,s} = \min(t, s)\sigma^2$.

Autocorrelations of RW

- Autocorrelations: $\rho_{t,s}$ = Cov(X_t, X_s)/sqrt(Var(X_t)sqrt(Var(X_s).
- Hence: $\rho_{t,s}$ = sqrt(t/s).
- Examples: $\rho_{1,2}$ = sqrt(1/2) = 0.707, $\rho_{8,9}$ = sqrt(8/9) = 0.943 and
  $\rho_{1,5}$ = sqrt(1/5) = 0.447. $\rho_{1,50}$ = sqrt(1/50) = 0.014.
- Notes:
  - 1) The values of X at neighbouring time points are more and more strongly correlated as time goes by, i.e., see $\rho_{1,2}$ vs. $\rho_{8,9}$.
  - 2) The values of X at distant time points are nearly uncorrelated, i.e., see $\rho_{1,5}$ vs. $\rho_{8,50}$.
- Consider: Cov(X_t, X_{t-k}) = (t – k)σ^2.
- Hence: ρ_k = sqrt[(t – k)/t].
- This is a very important result to detect non-stationarity.
- For the first few autocorrelations, the sample size t will be large relative to the number of autocorrelaions formed.
- For small values of k , the ratio [(t – k)/t] is approximately equal to unity.
- Thus, autocorrelations will show a slight tendency to decay.

An Example

- Consider an AR(2) process: $X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \varepsilon_t$
- If the autoregressive parameters satisfy stationarity conditions, i.e., $\varphi_1 + \varphi_2 < 1$ or $-\varphi_1 + \varphi_2 < 1$ and $|\varphi_2| < 1$ the process is stationary.
- If, however, $\varphi_1 + \varphi_2 = 1$, (unit autoregressive root) the process is not stationary and it becomes an ARIMA(1, 1, 0) process.
- To see this substitute for $\varphi_1 = 1 - \varphi_2$, then the process is written as; $X_t = (1 - \varphi_2)X_{t-1} + \varphi_2 X_{t-2} + \varepsilon_t$ ➔
  $X_t - X_{t-1} = - \varphi_2(X_{t-1} - X_{t-2}) + \varepsilon_t$
- This process is written as: $\Delta X_t = - \varphi_2 \Delta X_{t-1} + \varepsilon_t$.