Lectures on Economic Growth

Topic: The Basic Neoclassical (Solow) Model of Growth

an upper intermediate course offered at the 7th semester at the

Economics Department, University of Piraeus

Topics

Important questions to answer in this course:

- What are the sources of growth?

- The basic Solow model (w/o Technology)
  - Assumptions
  - Solving the basic Solow model
  - The Solow Diagram - Dynamics and Steady-State
  - Changes in the saving rate, s
  - The "Golden Rule" of capital accumulation

Back to the important question: What are the source(s) of growth?

Let's begin from the aggregate production function:

$Y = F(K, N)$
where K is physical capital and N is workers

We convert the aggregate production function in per worker terms (to be able to graph it and perform cross-country comparisons):
$\frac{Y}{N} = \frac{K}{N}$ [let's set: $\frac{Y}{N} = y$ & $\frac{K}{N} = \kappa$]

$y = f(k)$

The sources of growth:

- Capital accumulation, κ
- Technological progress, A (only in this case is constant and equal to A=1)

Physical capital

Capital in general is a productive asset that can be obtained from output. For this, you need to sacrifice consumption.

Do not think of capital as just machines in a factory. The factory itself, workers' desks, phones, pens, computers. A brand name.

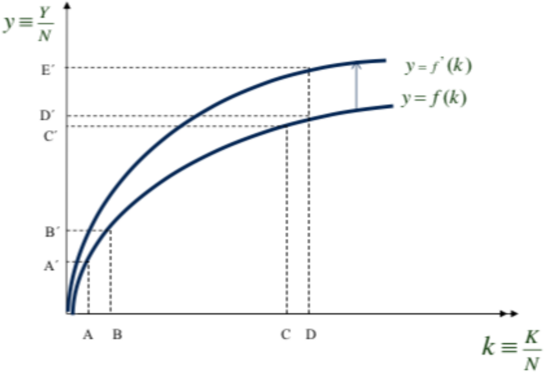
The nature of capital:

- Capital is productive.
- Capital is itself produced.
- Capital is rival in its use.
- Capital earns a return (as rent, or by being sold).
- Capital wears out.

Note that next to physical capital, human capital is a possible form.

Production-graph

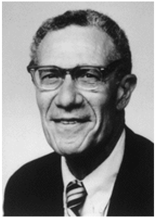
2.1 Production in per worker terms



Solow

Up to 1956, growth economists had used models in which labor and capital were complements. Solow (Nobel in economics, 1987) found:

> In that case, however, the possibility of steady growth would be a miraculous stroke of luck. Most economies, most of the time, would have no equilibrium growth path. The history of capitalist economies should be an alternation of long periods of worsening unemployment and long periods of worsening labor shortage.



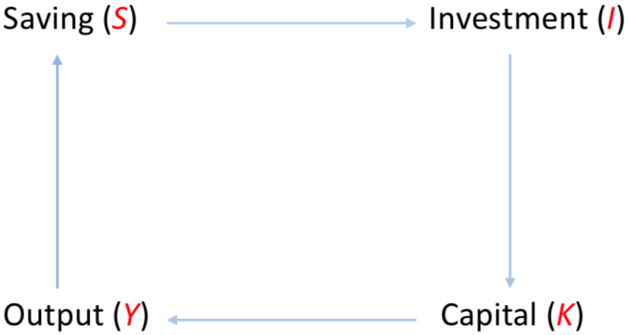
Source: nobelprize.org

Solow (cont.)

> I know that it occurred to me very early, as a natural-born macroeconomist, that even if technology itself is not so very flexible for each single good at a given time, aggregate factor intensity must be much more variable because the economy can choose to focus on capital-intensive or labor-intensive or land-intensive goods. Anyway, I found something interesting right away. (Solow's 1987 Nobel speech)

- Different combinations of labor and capital are possible
- Scarce factor has a high productivity, abundant factor has a low productivity
- Stable solution: when capital is scarce, it is very productive. This allows larger increases in capital than usual.

The basic macro aggregates



Developing a simple model of economic growth: Assumptions

- Countries produce and consume only a single and homogeneous good, "output", Y
- Closed economy (X=M=0) without Public Sector (T=G=0): Y = C + I or Y = C + S, so I=S
- Saving is a constant part of Income: S= sY, 0<s<1, I=sY
- Population (and labor, N) grows at a constant rate: $\mathrm{N(t) = N_0 * e^{nt}}$; $\frac{\dot{\mathrm{N}}}{\mathrm{N}} = \mathrm{n}\%$ [note: in other versions, the growth rate, n, is zero]
- Technology (A) is exogenous and not growing: g=0%

From capital stock equation to the Basic Solow Model

Suppose and economy produces with aggregate production (Cobb-Douglas): $\mathrm{Y} = \mathrm{K}^{\alpha}\mathrm{N}^{1-\alpha}$, which in per worker terms can be expressed as: $\mathrm{y} = \kappa^{\alpha}$ (1)

Capital stock evolution equation:

$\mathrm{K}_{t+1} = \mathrm{K}_t - \delta \mathrm{K}_t + \mathrm{I}_t$, but from the assumptions, I=S, an so we get:
$\mathrm{K}_{t+1} = \mathrm{K}_t - \delta \mathrm{K}_t + \mathrm{S}_t$, but from the assumptions, S=sY, and so we get:
$\mathrm{K}_{t+1} = \mathrm{K}_t - \delta \mathrm{K}_t + \mathrm{sY}_t$

Let's now express the capital stock evolution equation in per worker terms (to be able to compare different economies):
$\frac{\mathrm{K}_{t+1}}{\mathrm{N}_{t+1}} = \frac{\mathrm{K}_t}{\mathrm{N}_t} - \delta \frac{\mathrm{K}_t}{\mathrm{N}_t} + \mathrm{s}\frac{\mathrm{Y}_t}{\mathrm{N}_t} - \mathrm{n}\frac{\mathrm{K}_t}{\mathrm{N}_t}$

[let's set: $\frac{\mathrm{K}_{t+1}}{\mathrm{N}_{t+1}} = \kappa_{t+1}$; $\frac{\mathrm{K}_t}{\mathrm{N}_t} = \kappa_t$ and $\frac{\mathrm{Y}_t}{\mathrm{N}_t} = \mathrm{y}_t$]

$\kappa_{t+1} - \kappa_t = \mathrm{sy}_t - (\delta + \mathrm{n})\, \kappa_t$ (2)

One can replace: $\mathrm{y}_t = \kappa_t^{\alpha}$ from (1), and so we have:

The basic Solow model

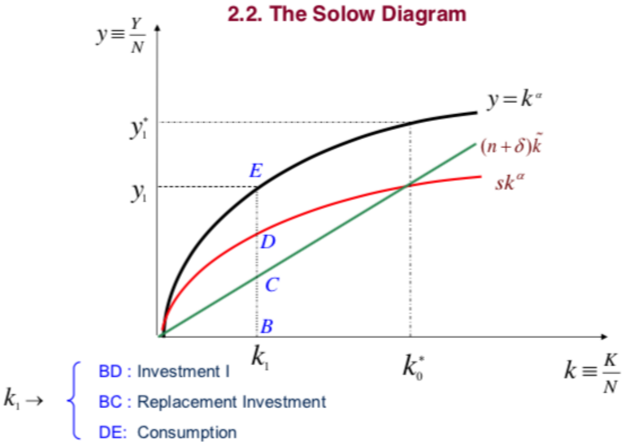
$$\underbrace{k_{t+1} - k_t}_{\text{Change in capital stock per capita}} = \underbrace{sk_t^\alpha}_{\text{Investment in t}} - \underbrace{(n+\delta)k_t}_{\text{"Replacement Investment" in t}}$$

- Capital per worker will increase (decrease) if total investment is larger (smaller) than replacement investment.
- Investment is a (less and less) increasing function of capital per worker, because of the properties of the $f(\cdot)$ function.
- Replacement investment is a proportional function of capital per worker.

Consumption per worker can be expressed as:

$$c_t = y_t - sy_t = (1-s)y_t = (1-s)k_t^\alpha$$

The basic Solow model - diagram



The basic Solow model - dynamics

Dynamics

Dynamics I: For capital per worker at k_1 , investment per worker (distance B-D in the graph) is larger than replacement investment per worker (distance B-C in the graph) so capital per worker will increase and also output per worker and consumption per worker (distance D-E in the graph for capital per worker at k_1).

$$sy > (n+\delta)k \Leftrightarrow Investment > Replacement\ I \Rightarrow \dot{k} > 0 \Rightarrow$$

$$\begin{cases} \rightarrow \uparrow y = f(\uparrow k) \Rightarrow \uparrow I = s(\uparrow y) \\ \rightarrow \uparrow ((\mathrm{n}+\delta) \uparrow \mathrm{k}) \end{cases}$$

Dynamics II: Given s and $f(\cdot)$, the economy converges in time to the point where total investment is equal to replacement investment. This happens at k_0^* in the graph above.

The basic Solow model - equilibrium

The Steady State

This "growth model" has an equilibrium point at k_0^* at which output, consumption and saving (all per worker) are well defined, unique and stable.

This equilibrium is a "steady state" of the economy where output and associated variables (consumption, saving and investment) **grow** at the same rate as population.

$$k = k^* \Rightarrow \frac{\dot{k}^*}{k^*} = 0 \qquad y = f(k) \Rightarrow \frac{\dot{y}^*}{y^*} = 0$$

$$k = \frac{K}{N} \text{ ; log and dt} \Rightarrow \frac{\dot{K}^*}{K^*} = \frac{\dot{N}^*}{N^*} = n$$

$$\frac{\dot{Y}^*}{Y^*} = \frac{\dot{N}^*}{N^*} = n$$

Technical notes on the growth rates

At the steady-state, the economy rests.

$\kappa_t = \frac{K_t}{N_t}$ and $y_t = \frac{Y_t}{N_t}$ grow at 0%

- N grows at n% (assumption)

- K grows also at n% . This is because $K_t = \kappa * N_t$.

  - take logs: $\ln(K_t) = \ln(\kappa) + \ln(N_t)$, then,
  - totally differentiate: $\frac{\dot{K}_t}{K_t} = \frac{\dot{\kappa}}{\kappa} + \frac{\dot{N}_t}{N_t}$ [note that dotted variables are derivates over time: $\dot{x}_t = \vartheta x_t / \vartheta t$]

Therefore, $\frac{\dot{K}_t}{K_t} = 0\% + n\% = n\%$

- Y also grows at n% (to show this, define Y as $Y_t = \kappa * N_t$ and follow the two-step process above)

The basic Solow model - equilibrium (cond.)

This steady state is characterized by k_0^* for which:

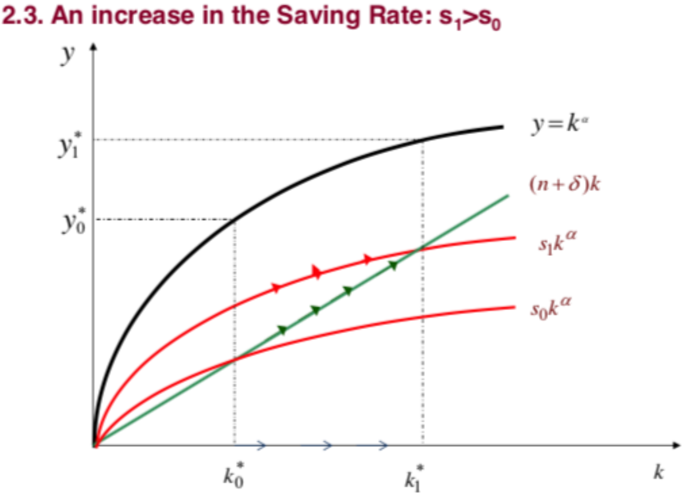
$$sy_0^* = (n+\delta)k_0^* \Rightarrow \dot{k}^* = 0 \quad \text{solving for } k_0^*:$$

$$k_0^* = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$y = k^\alpha \Rightarrow y_0^* = \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

$$c_0^* = (1-s)y_0^*$$

The basic Solow model - changes in saving rate, s



The basic Solow model - changes in s (cond.)

$$s_1 y_0^* > (n+\delta)k_0^* \Leftrightarrow Investment > Replacement\ I \Rightarrow \dot{k} > 0$$

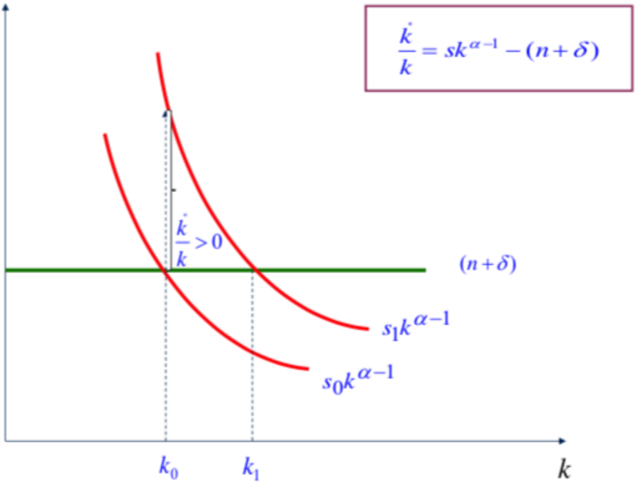
It is actually a saving rate "s_0" that, given "$f(\cdot)$", "n" and "δ", determines the steady state equilibrium of an economy k_0^*. What are the consequences of a change in the saving rate?

Few predictions can be obtained from our simplified Solow model:

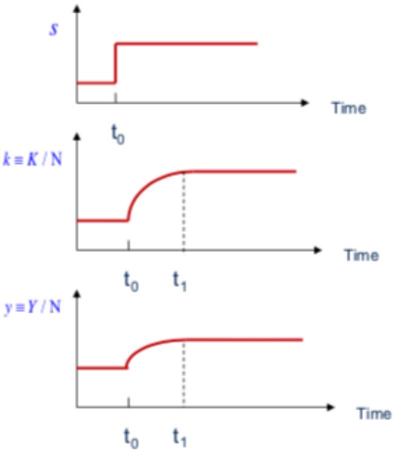
The level of steady state output per worker and capital per worker increase if the saving rate increases. $y_1^* > y_0^*$; $k_1^* > k_0^*$

But the saving rate does not affect the growth rate of output per worker (which is zero in the model so far) nor the output (which is n).

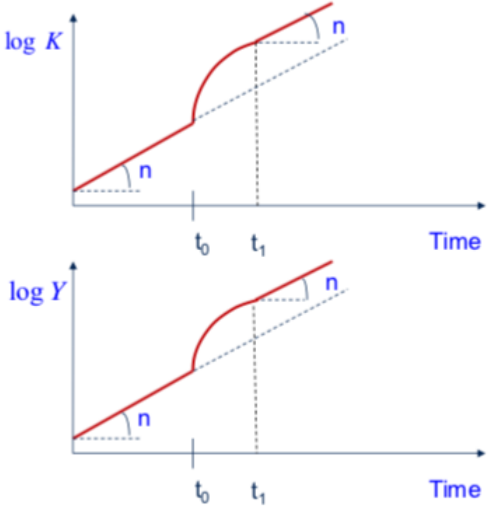
The basic Solow model - changes in s and adjustments



The basic Solow model - changes in s and adjustments (cond.)



The basic Solow model - changes in s and adjustments (cond.)



Changes of s: growth effects vs. level effects

From these experiments, we see that changes in s cause growth; but only transitory growth. In general, in the Solow model,

- all policy changes have level effects.

So a policy to increase s will cause growth for a while, but not for ever.

Optimality - who determines s?

The parameter s is given in the model. What determines it?
According to the book, many factors influence s: thriftiness, financial sector, institutions.
But we could ask ourselves: what is the optimal s?

Aside: for two reasons

- If people act optimally, thius will allow us to understand the economy: positive economics
- This tells you what people should actually do / should have done: normative economics

There exists a level of s that maximizes steady-state consumption. But this s is too high if we take account of time preference.

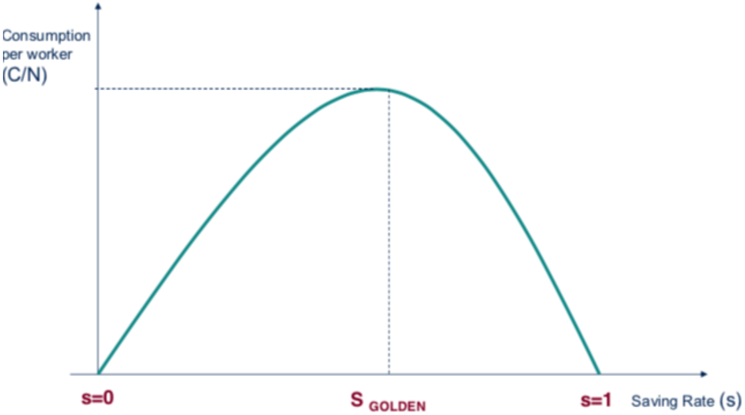
The basic Solow model - "Golden Rule" of capital accumulation

- Consumption per worker is the vertical distance between the production and the saving functions (see graph above). At every possible saving-determined steady state equilibrium value of K/N, there will correspond a unique steady state consumption per worker level and thus this variable can be represented as a function of the saving rate "s".

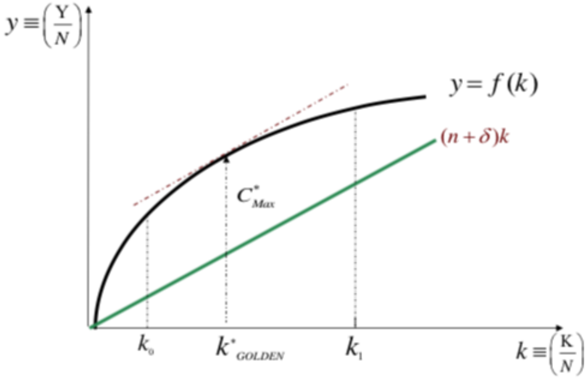
- The properties of the production function imply that steady state consumption per worker will increase with the saving rate up to a point beyond which it will start to decrease. The situation where steady state consumption per worker is maximum is called the "golden rule" and the corresponding saving rate and steady state capital per worker are called the "golden rule" saving rate and capital per worker.

-Governments can actually run surpluses (deficits) in order to help increase (decrease) the private saving ratio if it is too low (high) aiming at the golden rule capital per worker ratio in the economy and thus golden rule (i.e. maximum) consumption per worker.

The Golden Rule - saving rate



Golden Rule - consumption (graph)



Golden Rule consumption

Let's calculate where consumption per worker is maximized.
Define consumption per worker (c) as:

$\mathrm{c} = \mathrm{y} - \mathrm{sy} \Rightarrow \mathrm{sy} = \mathrm{y\text{-}c}$

From the Solow model:

κ_{t+1} - κ_t $= \mathrm{sy}_t$ - $(\delta + \mathrm{n})\ \kappa_t \Rightarrow$

κ_{t+1} - κ_t $= (\mathrm{y}_t - \mathrm{c}_t)$ - $(\delta + \mathrm{n})\ \kappa_t \Rightarrow$

κ_{t+1} - κ_t $= (\mathrm{f}(\kappa) - \mathrm{c}_t)$ - $(\delta + \mathrm{n})\ \kappa_t$

At the steady-state: κ_{t+1} - κ_t $= 0$

$\mathrm{c} = \mathrm{f}(\kappa) - (\mathrm{n} + \delta)$ Hence, one needs to maximaze w.r.t.:

MAX $\mathrm{c}* = \mathrm{f}(\kappa*) - (\mathrm{n} + \delta)$

w.r.t. $\kappa*$

F.O.C: $\frac{\vartheta \mathrm{c}*}{\vartheta \kappa*} = \mathrm{f}'(\kappa*) - (\mathrm{n} + \delta) = 0 \Rightarrow \mathrm{f}'(\kappa*) = \mathrm{n} + \delta$

S.O.C: $\frac{\vartheta^2 \mathrm{c}*}{\vartheta \kappa*_2} = \mathrm{f}'(\kappa*) < 0$

Golden Rule - dynamics

If $s_0 < s_{GOLDEN} \Rightarrow k_0^* < k_{GOLDEN}^* \Rightarrow f'(k_0^*) > (n+\delta)$

That suggests that increasing the saving rate you can increase –in steady state- the consumption per worker.

If $s_1 > s_{GOLDEN} \Rightarrow k_1^* > k_{GOLDEN}^* \Rightarrow f'(k_1^*) < (n+\delta)$

That suggests that decreasing the saving rate, will increase –in steady state- the consumption per worker.

In the next study section, we will introduce new evidence for this fact.

Uses of the Solow model

How much of income differences do differences in k explain?
Not very much. You need very large differences in k if they need to explain everything.

However, the model can be fixed with some extra theory. It also yields these predictions:

1. All else equal, poor countries grow faster than rich countries.
2. All else equal, countries that invest more grow faster.
3. An increase in investment leads to faster growth, for a while.

Note also: the model does not yet explain perpetual growth.