# Lectures on Economic Growth

## Topic: The Solow Model - adding Technology

an upper intermediate course offered at the 7th semester at the

Economics Department, University of Piraeus

# Topic

In previous lecture, technology was absent from the model.

In today's lecture, we introduce technology into the Solow model which grows at constant rate, g%.

However, the model does NOT explain HOW and WHO develops technology. Only that technology exists (level of technology is A) and grows at constant rate, g%.

This is why the Solow (or neoclassical) model of growth is also called "exogenous" model of growth, as technology is given to a country in an exogenous (not explained via a model) manner.

# Assumptions

- Countries produce and consume only a single and homogeneous good, "output", Y
- Closed economy (X=M=0) without Public Sector (T=G=0): Y = C + I or Y = C + S, so I=S
- Saving is a constant part of Income: S= sY, 0<s<1, I=sY
- Population (and labor, N) grows at a constant rate: $N(t) = N_0 e^{nt}$; $\frac{\dot{N}}{N} = n\%$
- Technology (A) grows at a constant rate: $A(t) = A_0 e^{gt}$; $\frac{\dot{A}}{A} = g\%$

# The Solow model with Technology

Suppose and economy produces with aggregate production (Cobb-Douglas): $Y = K_t^{\alpha}(A_t N_t)^{1-\alpha}$. Then, to express it in per "effective" worker terms, divide by AN, so one gets:

$\tilde{y}_t = \tilde{\kappa}_t^{\alpha}$ (1) where $\tilde{\kappa}_t = \frac{K_t}{A_t Nt}$

Capital stock evolution equation:

$K_{t+1} = K_t - \delta K_t + I_t$, but from the assumptions, I=S, an so we get:
$K_{t+1} = K_t - \delta K_t + S_t$, but from the assumptions, S=sY, and so we get:
$K_{t+1} = K_t - \delta K_t + sY_t$

Let's now express the capital stock evolution equation in per "effective" worker terms):

$\frac{K_{t+1}}{A_{t+1}N_{t+1}} = \frac{K_t}{A_t N_t} - \delta \frac{K_t}{A_t N_t} + s\frac{Y_t}{A_t N_t} - n\frac{K_t}{A_t N_t} - g\frac{K_t}{A_t N_t}$

[let's set: $\frac{K_{t+1}}{A_{t+1}N_{t+1}} = \tilde{\kappa}_{t+1}$; $\frac{K_t}{A_t N_t} = \tilde{\kappa}_t$ and $\frac{Y_t}{A_t N_t} = \tilde{y}_t$]

$\tilde{\kappa}_{t+1}$ - $\tilde{\kappa}_t = sy_t$ - $(\delta + n + g)\tilde{\kappa}_t$ (2)

One can replace: $\tilde{y}_t = \tilde{\kappa}_t^{\alpha}$ from (1), and so we have:

# The Solow model with technology

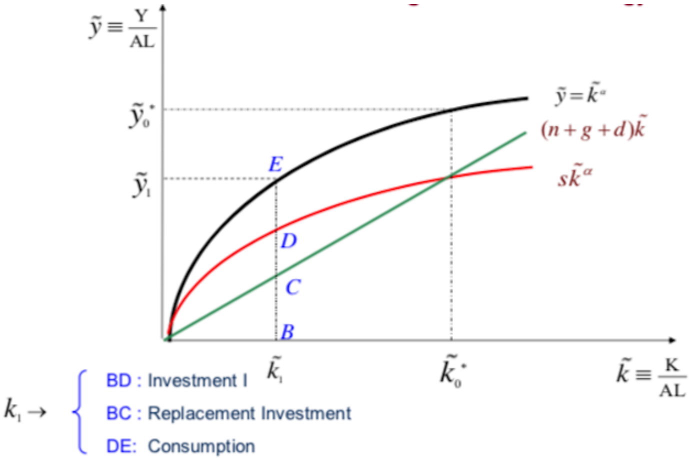
$$\underbrace{\tilde{k}_{t+1} - \tilde{k}_t}_{\text{Change in capital stock per capita}} = \underbrace{s\tilde{k}_t^{\alpha}}_{\text{Investment in t}} - \underbrace{(g + n + \delta)\tilde{k}_t}_{\text{"Replacement Investment" in t}}$$

- Capital-technology ratio will increase (decrease) if total investment is larger (smaller) than replacement investment (depreciation).
- Investment is a (less and less) increasing function of capital-technology ratio, because of the properties of the $f(\cdot)$ function.
- Replacement investment is a proportional function of capital-technology ratio .

Consumption per worker to technology can be expressed as:

$$\tilde{c}_t = \tilde{y}_t - s\tilde{y}_t = (1-s)\tilde{y}_t = (1-s)\tilde{k}_t^{\alpha}$$

# The Solow model with technology - The graph



# The basic Solow model - The dynamics

**Dynamics**

Dynamics I: For capital-technology ratio at $\tilde{k}_1$ investment per worker (distance B-D in the graph) is larger than replacement investment per worker (distance B-C in the graph) and capital-technology ratio will increase and thus will do output-technology ratio and consumption-technology ratio (distance D-E in the graph for capital per worker at $\tilde{k}_1$ )

$$s\tilde{y} > (g+n+\delta)\tilde{k} \Leftrightarrow Investiment > Replacement\ I \Rightarrow \dot{\tilde{k}} > 0 \Rightarrow$$

$$\begin{cases} \rightarrow\uparrow \tilde{y} = f(\uparrow \tilde{k}) \Rightarrow \uparrow I = s(\uparrow \tilde{y}) \\ \rightarrow\uparrow ((g+n+\delta)\uparrow \tilde{k}) \end{cases}$$

Dynamics II: Given $s$ and $f(\cdot)$, the economy converges with time to the point where total investment is equal to replacement investment. This happens at $\tilde{k}_0^*$ in the graph above.

# The basic Solow model - The equilibrium

**The Steady State**

This "growth model" has an equilibrium point at $\tilde{k}_0^*$ at which output, consumption and saving (all per worker to technology) are well defined, unique and stable.

The new *state* variables will be: $\tilde{k} \equiv \frac{K}{AN}$ ; $\tilde{y} \equiv \frac{Y}{AN}$

This steady state is characterized by $\tilde{k}_0^*$ for which: $\tilde{k} = \tilde{k}_0^* \Rightarrow \frac{\dot{\tilde{k}}^*}{\tilde{k}^*} = 0 \Rightarrow$

$$y_0^* = k_0^{*\alpha} \Rightarrow \frac{\dot{\tilde{y}}^*}{\tilde{y}^*} = 0$$

$$s\tilde{y}_0^* = (g + n + \delta)\tilde{k}_0^* \quad \text{solving for } \tilde{k}_0^* :$$

$$\tilde{k}_0^* = \left(\frac{s}{n+g+d}\right)^{\frac{1}{1-\alpha}} \qquad \tilde{y}_0^* = \left(\frac{s}{n+g+d}\right)^{\frac{\alpha}{1-\alpha}} \qquad \tilde{c}_0^* = (1-s)\tilde{y}_0^*$$

# The basic Solow model - Equilibrium (cond.)

The level of output per worker is:

$$y^*(t) = A(t)\left(\frac{s}{g+n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Note the dependence of y and A on time.
Output per worker along the steady state the balanced growth path is determined by technology, the investment rate and the population growth rate.
For the special case of A=1 and g=0 this result is identical to that derived in the basic Solow model.

# At the steady-state: what is the growth rate of the variables?

At the steady-state, the economy rests.

$\kappa_t = \frac{K_t}{A_t N_t}$ and $y_t = \frac{Y_t}{A_t N_t}$ grow at 0%

- N grows at n% (by assumption)

- A grows at g% (by assumption)

- K grows also at n% + g% . This is because $K_t = \kappa * A_t * N_t$.
  - take logs: $\ln(K_t) = \ln(\kappa) + \ln(N_t) + \ln(A_t)$, then,
  - totally differentiate: $\frac{\dot{K}_t}{K_t} = \frac{\dot{\kappa}}{\kappa} + \frac{\dot{N}_t}{N_t} + frac\dot{A}_t A_t$ [note that dotted variables are derivates over time: $\dot{x}_t = \vartheta x_t / \vartheta t$]

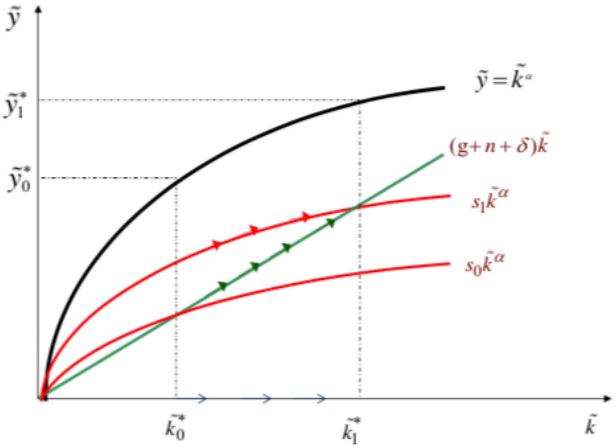
Therefore, $\frac{\dot{K}_t}{K_t} = 0\% + n\% + g\% = n\% + g\%$

- Y also grows at n% + g% ( to show this, define Y as $Y_t = y * A_t * N_t$ and follow the two-step process above)

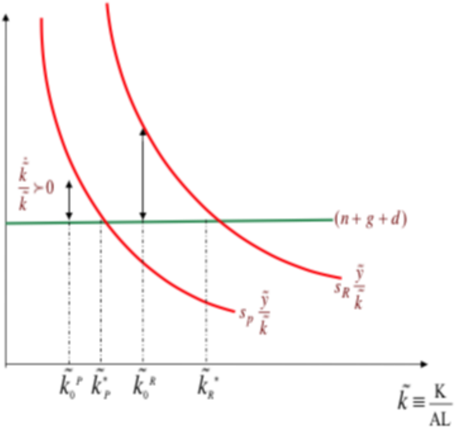
# Solow model versions - a comparison

| VARIABLES | Growth Rate (With A) | Growth Rate (Without A) |
|---|---|---|
| $\tilde{k} \equiv \frac{K}{AL}$ | 0 | - |
| $\tilde{y} \equiv \frac{Y}{AL}$ | 0 | - |
| K/L | g | 0 |
| Y/L | g | 0 |
| A | g | - |
| L | n | n |
| K | g+n | n |
| Y | g+n | n |
| AL | g+n | - |

# Solow model with Technology - changes in saving rate, s



# Solow model with Technology - changes in s and adjustments



# Solow model with Technology - changes in s and adjustments

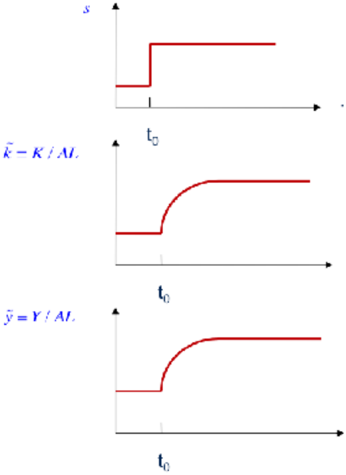
$\tilde{y} = f(\tilde{k}) = \tilde{k}^{\alpha}$ *Taking logs and derivatives with respect to time*:

$$\frac{\dot{\tilde{y}}}{\tilde{y}} = g + \alpha \frac{\dot{\tilde{k}}}{\tilde{k}}$$

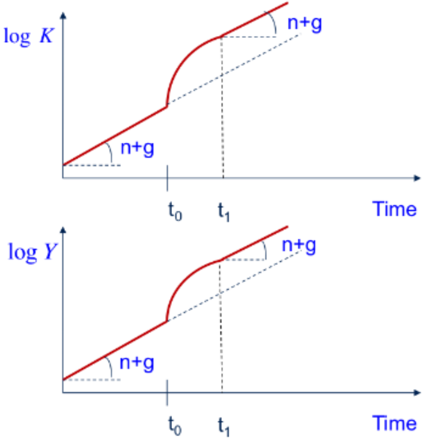
As $\dot{\tilde{k}}/\tilde{k}$ is positive and decreasing during transition and zero in the Steady State, the growth rate of output per worker to technology $\dot{\tilde{y}}/\tilde{y}$ increases and then it´s decreasing until the new Steady State.

We can represent the *transition path* of the variables in the following graphs:

# Solow model with Technology - changes in s and adjustments (cond.)



# Solow model with Technology - changes in s and adjustments (cond.)



# Solow model with Technology - the Golden Rule of Capital

$$\tilde{c} = \tilde{y} - s\,\tilde{y} \Rightarrow s\,\tilde{y} = \tilde{y} - \tilde{c}$$

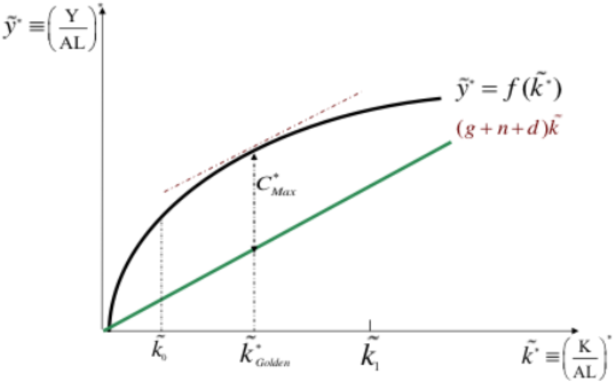
$$\dot{\tilde{k}} = (\tilde{y} - \tilde{c}) - (g+n+\delta)\,\tilde{k} \Rightarrow \dot{\tilde{k}} = 0 \;;\; \text{s.s.} \Rightarrow \quad \tilde{c}^{*} = f(\tilde{k}^{*}) - (g+n+\delta)\,\tilde{k}^{*}$$

$$\underset{\{\tilde{k}^{*}\}}{MAX.}\ \tilde{c}^{*} = f(\tilde{k}^{*}) - (g+n+\delta)\,\tilde{k}^{*}$$

F.O.C: $\dfrac{\partial \tilde{c}^{*}}{\partial \tilde{k}^{*}} = f'(\tilde{k}^{*}_{GOLDEN}) - (g+n+\delta) = 0 \Rightarrow f'(\tilde{k}^{*}_{GOLDEN}) = (g+n+\delta)$

S.O.C.: $\dfrac{\partial^{2} \tilde{c}^{*}}{\partial \tilde{k}^{*2}} = f''(\tilde{k}^{*}_{GOLDEN}) < 0$

# Golden Rule - graph



# Golden Rule - dynamics

If $s_0 < s_{GOLDEN} \Rightarrow \tilde{k}_0^* < \tilde{k}_{GOLDEN}^* \Rightarrow f'(\tilde{k}_0^*) > (g+n+\delta)$

That suggests that increasing the saving rate you can increase –in steady state- the consumption per worker to technology.

If $s_1 > s_{GOLDEN} \Rightarrow \tilde{k}_1^* > \tilde{k}_{GOLDEN}^* \Rightarrow f'(\tilde{k}_1^*) < (g+n+\delta)$

That suggests that decreasing the saving rate, will increase –in steady state- the consumption per worker to technology.

# Evaluation of the Solow model

1. Why are some countries rich and others are poor?
   - Because >s; <n; >A(t)

2. Why do economies exhibit sustained growth in the Solow model?
   - Because of technological progress.

$$\frac{\dot{y}^*}{y^*} = g$$

3. Why do some countries grow more quickly than others? How does the Solow model account for differences in growth rates across countries?
   - Because of the differences in technological progress.
   - Because of the transition dynamics.

$$\frac{\dot{y}}{y} = g + \alpha \frac{\dot{k}}{k}$$

# Two countries

Suppose we are looking at two countries with different levels of GDP/capita. Both are experiencing transitory growth towards the steady state.

Suppose these countries are equal, in the sense that

- They have the same production function.
- They have the same rate of investment s.
- They have the same rate of depreciation $\delta$.

Then they will eventually end up at the same level of GDP/capita: this is convergence.

# Convergence in practice

Automatic convergence would be pretty nice, if it really occurs. When we have a number of time series for GDP/capita, how can we check?

- $\sigma$-convergence: the variance of GDP/capita across countries decreases.
- $\beta$-convergence: countries grow faster when their initial GDP/capita is low.

In the standard Solow model, we expect to see both kinds of convergence. When there are random, independent, shocks to each economy, we may not see $\sigma$-convergence, but we still expect to see $\beta$-convergence.

# Measuring convergence

Measuring $\sigma$-convergence is straightforward: just look at the variance of GDP/capita over time.

| | 1950 | 1960 |
|---|---|---|
| France | 4151 | 5981 |
| Italy | 2770 | 4660 |
| Average | 3461 | 5321 |
| SD | 563 | 539 |

This kind of convergence is pretty rare: only found in homogenous groups of countries or regions.

# Measuring convergence (cont.)

Measuring $\beta$-convergence can be done with a cross-section regression across a group of countries:

$$\mathrm{growth_i} = \mathrm{c} + \beta \cdot \log \mathrm{y_0}$$

For instance

| | GDP 1950 | growth 1950-1990 |
|---|---|---|
| Belgium | 4447 | 3.055 |
| France | 4151 | 3.356 |
| Germany | 3467 | 4.179 |
| Luxemburg | 6850 | 2.394 |
| Netherlands | 4531 | 2.838 |

| | Coefficient | Std. Error | t-Statistic |
|---|---|---|---|
| C | 23.52887 | 5.470734 | 4.300861 |
| beta | -5.56465 | 1.494368 | -3.723751 |

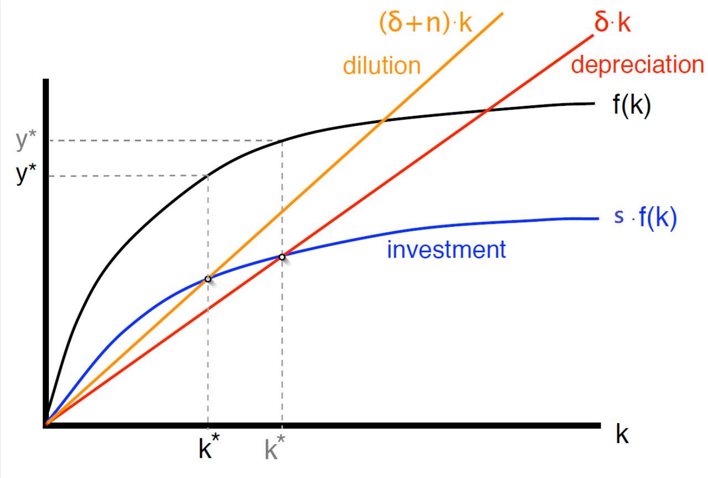
# Different economies

But what if two (or more) economies are not the same at all?

They can have different values for s, $\delta$. Or, even, different production functions.

If you have read the 'parable' of chapter 2, you know what could cause these differences in the production function: institutions, culture, policy.

# Non-convergence




Different steady states: country 1 ends up somewhere else than country 2.

# Different economies

When the steady states are different, we do not expect $\sigma$- or $\beta$-convergence.

The one thing we can hope for is

- Conditional $\beta$-convergence: countries grow faster when their initial GDP/capita is low, conditional on the level of their steady state.

So in the previous growth regression, we need to insert regressors that proxy for the steady state.

Note that

$$\sigma\text{-convergence} \Rightarrow \beta\text{-convergence} \Rightarrow \text{cond. } \beta\text{-convergence}$$

but not the other way around.

# What to expect?

Based on theory, we do not expect $\sigma$- and $\beta$-convergence except in lucky circumstances.

However, we should be able to find conditional $\beta$-convergence, provided we are able to find determinants of the steady state.

| | | |
|---|---|---|
| | $\sigma$-conv. | No, except in clubs |
| Results: | $\beta$-conv. | Sometimes in regions |
| | cond. $\beta$ | Mostly successful |

# Growth regressions

Barro and Sala-i-Martin (1995), estimate the following regression:

$$(\mathrm{g_y})_\mathrm{i} = \mathrm{x} + \mathrm{a} \cdot \log((\mathrm{y_0})_\mathrm{i}) + \mathrm{bX_i} + \epsilon_\mathrm{i}$$

for a cross-section of countries, with a set of instruments X

- Expect a to be negative
- Look at x for average technological progress, but only when your regressors have zero means.
- Look at the realization of the error $\epsilon_\mathrm{i}$ for the unexplained growth (or non-growth).