UTILITY THEORY – STOCHASTIC DOMINANCE

Individuals prefer more wealth to less, so the utility function is upwards sloping:

$U'(x) > 0$

Individuals exhibit risk aversion (they require a premium in order to accept risk). We can show that this implies a utility function that is convex (or concave downwards), increasing thus at a decreasing rate:

$U''(x) < 0$

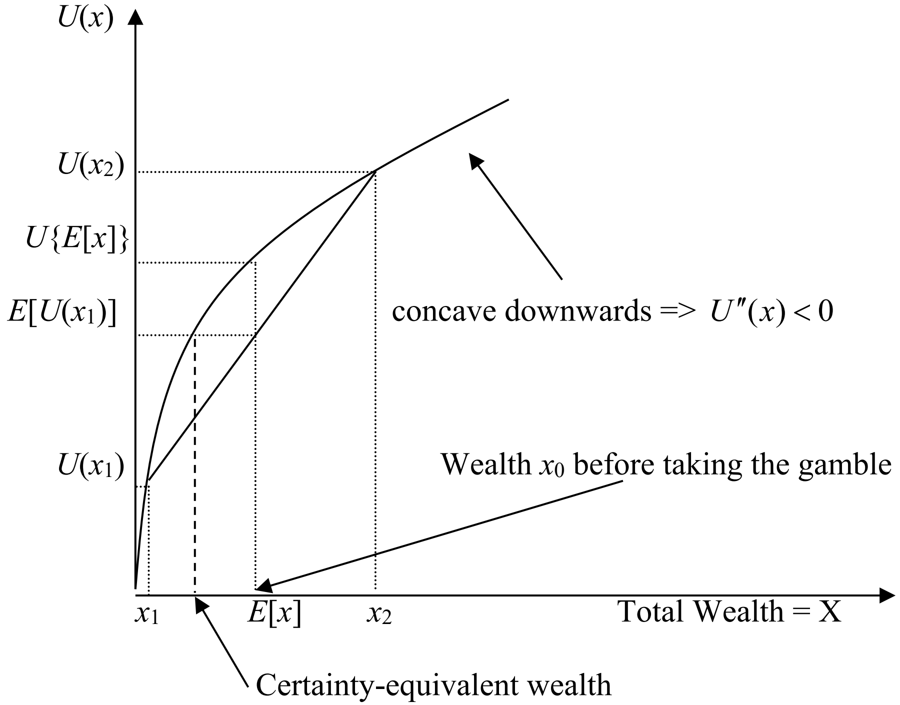
Individuals exhibit decreasing absolute risk aversion (same absolute level of risk is more tolerable at higher levels of wealth). This provides the less intuitive result (see discussion on work of Pratt):

$U'''(x) > 0$

In general, for risk aversion: $U\{E[x]\} > E[U(x)]$ => $U''(x) < 0$

for risk-neutrality: $U\{E[x]\} = E[U(x)]$ => $U''(x) = 0$

and for risk-loving: $U\{E[x]\} < E[U(x)]$ => $U''(x) > 0$


$U(x)$
$U(x_2)$
$U\{E[x]\}$
$E[U(x_1)]$
$U(x_1)$
concave downwards => $U''(x) < 0$
Wealth x_0 before taking the gamble
x_1
$E[x]$
x_2
Total Wealth = X
Certainty-equivalent wealth

Definitions

The Markowitz risk premium:

Risk Premium: $E[x]$ – (certainty-equivalent wealth) > 0 for risk-aversion

Cost of the gamble: x_0 – (certainty-equivalent wealth)

If **Cost of the gamble** < 0 we are willing to pay in order to get the gamble, else we want to be compensated in order to accept it.

Actuarially neutral gamble: when $x_0 = E[x]$ in which case the **Markowitz risk premium** equals the **Cost of the gamble**.

Example 1:

Suppose that we are faced with the following gamble: we have a current level of wealth of 10, and we have a logarithmic utility function. There is a 80% probability that we ending up with 5 (a decline of 5) and a 20% probability that we ending up with 30 (an increase of 20). The expected utility of the gamble is

$$E[U(W)]=0.8U(5)+0.2U(30)=0.8(1.61)+0.2(3.4)=1.97$$

From the logarithmic utility function we see that the level of wealth that provides 1.97 utiles is 7.17. On the other hand, the expected level of wealth, if we accept the gamble is 10, equal to the current level of wealth.

$$E(W)=0.8(5)+0.2(30)=10$$

Therefore, given the logarithmic utility function, we will be willing to pay up to 2.83 in order to avoid the gamble. If we are offered insurance against the gamble that costs less that 2.83, we will buy it. This is the Cost of the gamble.

Example 2:

A risk averse individual has the same logarithmic utility function, and the same current level of wealth, that is 10, but the gamble is 10% chance of winning 10 and 90% chance of winning 100. We can easily compute the following numbers:

Current Wealth = 10

Expected wealth = 101

Certainty Equivalent Wealth = 92.76

Risk Premium = Expected wealth – Certainty Equivalent Wealth = 101-92.76=8.24

Since the gamble is favorable, we would be willing to pay a positive amount to take the gamble. The cost of the gamble is

Cost of the gamble=Current Wealth – Certainty Equivalent = 10-92.76=-82.76

In other words, we are willing to pay 82.76 in order to take the gamble.

Example:

If you are exposed to a 50/50 chance of gaining of losing 1.000 and insurance that removes the risk costs 500, at what level of wealth will you be indifferent relative to taking the gamble ot paying the insurance? That is, what is your certainty equivalent wealth? Assume your utility function is $U(W)=-W^{-1}$.

The utility function is

$$U(W) = -W^{-1}$$

Therefore, the level of wealth corresponding to any utility is

$$W = -(U(W))^{-1}$$

Therefore, the certainty equivalent wealth for a gamble of $\pm 1,000$ is $\bar{W}$.

$$\bar{W} = -[.5(-(W+1,000)^{-1}) + .5(-(W-1,000)^{-1})]^{-1}$$

The point of indifference will occur where your current level of wealth, W, minus the certainty equivalent level of wealth for the gamble is just equal to the cost of the insurance, $500.

Thus, we have the condition

$$W - \bar{W} = 500$$

$$W - \left[-\frac{1}{.5\frac{-1}{W+1,000} + .5\frac{-1}{W-1,000}} \right] = 500$$

$$W - \left[\frac{-1}{\frac{-W}{W^2 = 1,000,000}} \right] = 500$$

$$W + \left[\frac{-W^2 - 1,000,000}{-W} \right] = 500$$

$$W^2 - W^2 + 1,000,000 = 500W$$

$$W = 2,000$$

Therefore, if your current level of wealth is \$2,000, you will be indifferent. Below that level of wealth you will pay for the insurance while for higher levels of wealth you will not.

RISK AVERSION AND PRATT´S APPROXIMATION

OF THE RISK PREMIUM

We assume the existence of a small gamble z that is actuarially neutral, so that its expectation equals zero, $E\left[z\right]=0$. If it is not actuarially neutral, $E\left[z\right]=\mu$, we can make it: Assume the new gamble $z'=z-\mu$ and the new starting wealth that now equals $x_0+\mu$. The utility of this gamble equals $U\left(x_0+z\right)=U\left(\left(x_0+\mu\right)+z'\right)$ and the gamble z is replaced by the actuarially neutral z'.

Pratt finds an approximation to the risk premium, π, so that

$U\left\{E\left[x_0+z-\pi\right]\right\}=E\left[U\left(x_0+z\right)\right]$ or equivalently $U\left\{x_0+E\left[z\right]-\pi\right\}=E\left[U\left(x_0+z\right)\right]$. Stated in words he approximates the risk premium so that the utility of a certainty-equivalent equals the expected utility of wealth. Note that now $E\left[z\right]=0$.

He takes the Taylor series expansion of both sides (LHS and RHS) of

LHS = $U\left\{x_0+E\left[z\right]-\pi\right\}=E\left[U\left(x_0+z\right)\right]$= RHS.

Taylor series expansion of a function:

$$f(a+h)=f(a)+f'(a)h+\frac{f''(a)h^2}{2!}+\frac{f'''(a)h^3}{3!}+\ldots$$

Start with the LHS first:

$$U\left\{x_0+E[z]-\pi\right\}=U\left\{x_0-\pi\right\}$$
$$=U\left(x_0\right)-\pi U'\left(x_0\right)+\pi^2 U''\left(x_0\right)/2!-\pi^3 U'''\left(x_0\right)/3!+\ldots$$

but the premium must be small compared with initial wealth, so higher order terms are ignored to get $U\left\{x_0-\pi\right\}=U\left(x_0\right)-\pi U'\left(x_0\right)$.

We proceed similarly with the RHS expanding the term inside the expectation operator:

$$
\begin{aligned}
&E\left[U\left(x_0+z\right)\right]=E\left[U\left(x_0\right)+zU'\left(x_0\right)+z^2U''\left(x_0\right)/2!+z^3U'''\left(x_0\right)/3!+\ldots\right] \\
&=E\left[U\left(x_0\right)\right]+E\left[zU'\left(x_0\right)\right]+E\left[z^2U''\left(x_0\right)/2!\right]+E\left[z^3U'''\left(x_0\right)/3!\right]+\ldots \\
&=U\left(x_0\right)+0+\sigma_z^2U''\left(x_0\right)/2!+\sigma_z^3U'''\left(x_0\right)/3!+\ldots \\
&=U\left(x_0\right)+0+\sigma_z^2U''\left(x_0\right)/2!+0+0+\ldots
\end{aligned}
$$

since

$$
E\left[z^2U''\left(x_0\right)/2!\right]=E\left[z^2\right]U''\left(x_0\right)/2!=E\left[z-E\left[z\right]\right]^2U''\left(x_0\right)/2!=\sigma_z^2U''\left(x_0\right)/2!
$$

and thus, equating the two expansions we get

$$
\begin{aligned}
&U\left(x_0\right)-\pi U'\left(x_0\right)=U\left(x_0\right)+\sigma_z^2U''\left(x_0\right)/2!\Rightarrow \\
&\pi\left(x_0,z\right)=\pi=-0.5\sigma_z^2U''\left(x_0\right)/U'\left(x_0\right)\Rightarrow \\
&\pi=0.5\sigma_z^2r\left(x_0\right)
\end{aligned}
$$

where

$$
r\left(x_0\right)=-U''\left(x_0\right)/U'\left(x_0\right)\equiv-d\left[\ln U'\left(x_0\right)\right]/dx_0
$$

and $r\left(x_0\right)$ is a measure of local absolute risk aversion and can be interpreted as a measure of concavity. Pratt demonstrated similar results also for discrete distributions.

He demonstrates that for any two $U_1(x)$, $U_2(x)$, if $r_1(x) > r_2(x)$ for every x

then $\pi_1(x,z) > \pi_2(x,z)$ for every z (and $U_1(x)$ is globally more risk averse). He also demonstrates the equivalence between decreasing $r(x)$ and decreasing $\pi(x,z)$.

Notes.

Risk aversion implies concavity of $U(x)$, so for $\pi > 0 \Rightarrow U''(x) < 0$.

We shall define the measure of absolute risk aversion (ARA):

$$ARA = -\frac{U''(x)}{U'(x)}$$

It is called absolute risk aversion because it measures risk aversion for a given level of wealth.

We can multiply the measure of absolute risk aversion by the level of wealth to obtain what is known as the relative risk aversion (RRA)

$$RRA = -x\frac{U''(x)}{U'(x)}$$

Constant relative risk aversion implies that an individual will have constant risk aversion to a proportional loss of wealth even though the absolute loss increases as wealth does.

Decreasing absolute risk aversion (DARA) is a desirable property of $U(x)$, and this happens when $U'''(x) > 0$:

$$0 > r'(x) = \left[-\frac{U''(x)}{U'(x)}\right]' = -\frac{U'''(x)}{U'(x)} - (-1)U''(x)\frac{U''(x)}{\left[U'(x)\right]^2} = -\frac{U'''(x)}{U'(x)} + \left[\frac{U''(x)}{U'(x)}\right]^2$$

which holds if $U'''(x) > 0$ since $U'(x) > 0$.

Pratt also discusses conditions so that transforms of utility functions with DARA properties will provide a utility function with the same properties:

If $U(x)$ is DARA then $U(ax+b)$ is also DARA.

If $U_1(x)$ is DARA and $U_2(x)$ is also DARA, then $U_2[U_1(x)]$ is also DARA.

If $U_1(x)$, $U_2(x)$, $U_3(x)$,…$U_n(x)$ are DARA, then $\sum_{i=1}^{n} c_i U_i(x)$ with constants c_i is DARA.

Example. Does $U(x) = -(b-x)^c$, $x \le b$, $c>1$, possess the desirable properties?

$U'(x) = -c(b-x)^{c-1}(-1) > 0$ so it is increasing in wealth

$U''(x) = c(c-1)(b-x)^{c-2}(-1) < 0$ so it exhibits risk aversion

$r(x) = -\dfrac{U''(x)}{U'(x)} = -\dfrac{c(c-1)(b-x)^{c-2}(-1)}{-c(b-x)^{c-1}(-1)} = \dfrac{c-1}{b-x}$ so $r(x)$ increasing in x, not DARA.

For $c = 2$ and $b = \beta/(2\gamma)$ we have $-(b-x)^2 = -b^2 + 2bx - x^2$, multiply by constant γ

=> $-b^2\gamma + 2b\gamma x - \gamma x^2 = \alpha + \beta x - \gamma x^2$ so quadratic utility function is not DARA.

DARA are the logarithmic utility function, sums of negative exponentials, and the power function (and sums): $U(x) = -(x+d)^q$ with $d \ge 0$, $q > 0$.

The quadratic utility has the properties:

$$ARA = -\frac{U''(x)}{U'(x)} = -\frac{2b}{a-2bx}, \qquad \frac{d(ARA)}{d(x)} > 0$$

$$RRA = -\frac{U''(x)}{U'(x)} = -\frac{2b}{(a/x)-2b}, \qquad \frac{d(RRA)}{d(x)} > 0$$

Unfortunately, the quadratic utility function exhibits increasing ARA and increasing RRA. Neither of these properties makes sense intuitively. For example, an individual with increasing RRA would become more averse to a given percentage loss in wealth as wealth increases. A billionaire who loses half his wealth would lose more utility that the same person who started with 20 and ended up with 10. This is not intuitive.

Pratt – on proportional risk aversion

Consider a proportional risk $x_0(1+z)$ and define a new (proportional) risk premium π^* so that

$$U\{x_0 + E[x_0 z] - x_0\pi^*\} = E[U(x_0 + x_0 z)].$$

We can approximate again this new risk premium by taking like before the Taylor series expansion:

LHS = $U(x_0) - x_0\pi^* U'(x_0) + 0...$

and

RHS = $E\left[U(x_0) + x_0 z U'(x_0) + x_0{}^2 z^2 U''(x_0)/2 + ...\right]$

$\quad = U(x_0) + 0 + x_0{}^2\sigma_z^2 U''(x_0)/2 + ...$

so we get

$$\pi^*(x_0, z) = \pi^* = -0.5 x_0 \sigma_z^2 U''(x_0) / U'(x_0) = 0.5 \sigma_z^2 r^*(x_0)$$

with

$$r^*(x_0) = -x_0 U''(x_0) / U'(x_0) = x_0 r(x_0)$$

Pratt shows the equivalence between decreasing $r^*(x_0)$ and decreasing $\pi^*(x_0)$.

Should $r^*(x_0)$ be decreasing? This is debated, but he refers to some empirical evidence that it might be constant…

STOCHASTIC DOMINANCE

Stochastic dominance (SD) is a generalization of utility theory that eliminates the need to explicitly specify an individual's utility function. The main attractiveness of the SD approach is that it is nonparametric, in the sense that its criteria do not impose explicit specifications of an investor preferences or restrictions on the functional forms of probability distributions. Rather, they rely only on general preference and beliefs assumptions. Thus, no specification is made about the return distribution, and the observed discrete distribution is posited to represent the underlying distribution.

An asset stochastically dominates another if it provides greater wealth in every ordered state of nature (1st order stochastic dominance). We are given the cumulative density functions $F_A(W)$ and $F_B(W)$ of wealth W that result from two assets A and B, so $F = \int_{-\infty}^{k} f\left(W\right)dW$ is the cumulative probability that $W \leq k$. Mathematically, we obtain 1st order stochastic dominance of A over B, when

$[F_B(W) - F_A(W)] >= 0$ holds for all wealth levels.

We know that A is preferred to B if and only if terminal wealth satisfies that

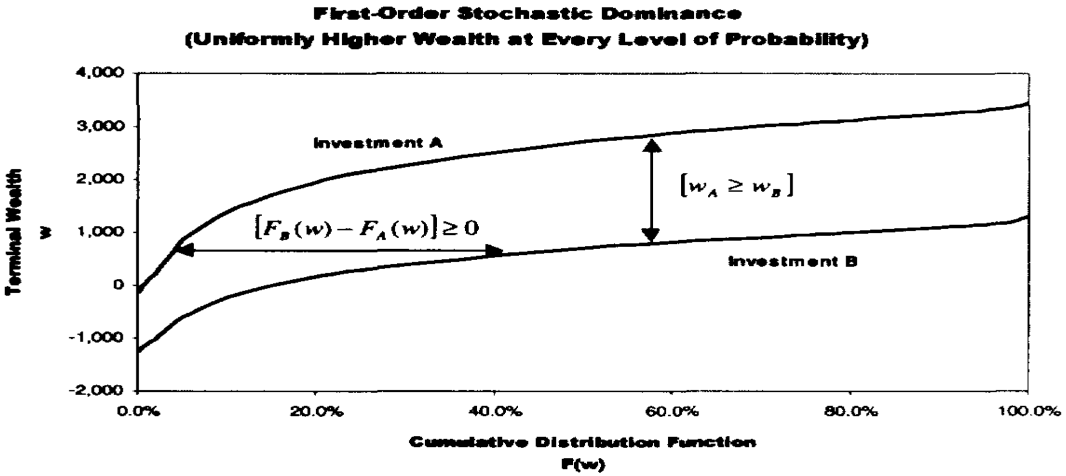
$$E_W[U(W_A) - U(W_B)] >= 0 \qquad (1)$$

We can integrate equation 1 by parts to yield:

$$\int_{-\infty}^{\infty} U(t) f_A(t)dt - \int_{-\infty}^{\infty} U(t) f_B(t)dt \geq 0 \Rightarrow$$
$$\int_{-\infty}^{\infty} U(t)[f_A(t) - f_B(t)]dt \geq 0 \Rightarrow$$
$$U(t)[F_A(t) - F_B(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} [F_A(t) - F_B(t)]U'(t)dt \geq 0$$
$$\int_{-\infty}^{\infty} [F_B(t) - F_A(t)]U'(t)dt \geq 0$$

But we know that U'(W)>0, so A dominates B by first order stochastic dominance if and only if $[F_B(W) - F_A(W)] >= 0$

First order stochastic dominance guarantees that for every increasing utility function asset *A* is preferred to asset *B* because it provides higher expected utility of wealth.



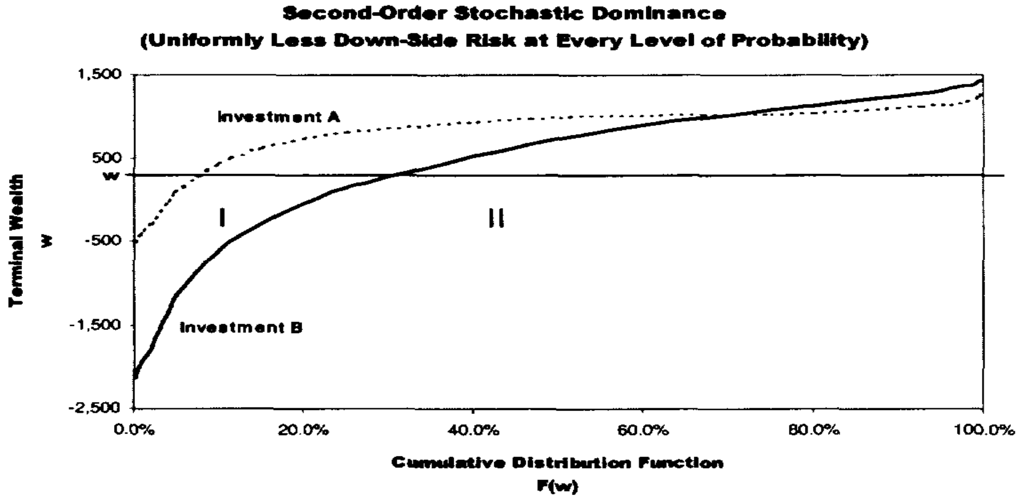
This figure depicts the cumulative distribution functions for two investments A and B that satisfy Equation 1. From this graph we can see that first-order stochastic dominance is equivalent to uniformly higher terminal wealth at every level of probability. In other words, the cumulative probability distribution (defined on wealth, W) for A always lies to the left of the cumulative distribution for B. Accordingly, first-order stochastic dominance is a weak result; rarely will an individual be faced with such an obvious investment choice. The weakness of this result arises from the fact that first-order stochastic dominance results from a weak utility function constraint, increasing wealth preference.

The FSD criterion places on the form of the function beyond the usual requirement that it is nondecreasing, i.e., investors prefer more to less. Thus, this criterion is appropriate for both risk averters and risk lovers since the utility function may contain concave as well as convex segments. Owing to its generality, the FSD permits a preliminary screening of investment alternatives eliminating those which no rational investor will ever choose. The SSD criterion adds the assumption of global risk aversion. This criterion is based on a stronger assumption and therefore, it permits a more sensible selection of investments.

For second order stochastic dominance we need utility functions to be not only increasing but also concave (exhibiting risk aversion). Then we obtain 2[st] order stochastic dominance of *A* over *B* (A provides higher expected utility of wealth – see proof in the book), when

$$\int_{-\infty}^{W_i}[F_B(W)-F_A(W)]dW \geq 0\text{, holds for all W}$$

$$F_A(W_i) \neq F_B(W_i) \text{ holds for some } W_i$$



This means that in order for investment A to dominate investment B for all risk averse investors, the accumulated area under the cumulative probability distribution of B must be greater that the accumulated area for A, below any given level of wealth. This implies that, unlike first-order order stochastic dominance, the cumulative density functions can cross.

We use 2nd order stochastic dominance when 1st order does not apply.

Example.

You have estimated the following probabilities for earnings per share of companies A and B:

p_i	Co. A	Co. B
.1	0	−.50
.2	.50	−.25
.4	1.00	1.50
.2	2.00	3.00
.1	3.00	4.00

(a) Calculate the mean and variance of the earnings per share for each company

(b) Explain how some investors might choose A and others might choose B if preferences are based on mean and variance

(c) Compare A and B, using the second-order stochastic dominance efficiency criterion.

(a)

p_i	Co. A	Co. B	$p_i A_i$	$p_i [A - E(A)]^2$	$p_i B_i$	$p_i[B - E(B)]^2$
.1	0	−.50	0	.144	−.05	.4000
.2	.50	−.25	.10	.098	−.05	.6125
.4	1.00	1.50	.40	.016	.60	0
.2	2.00	3.00	.40	.128	.60	.4500
.1	3.00	4.00	.30	.324	.40	.6250
			1.20	.710	1.50	2.0875

$$E(A) = 1.20, \quad \sigma_A = .84$$
$$E(B) = 1.50, \quad \sigma_B = 1.44$$

(b) Figure S3.7 shows that a risk averse investor with indifference curves like #1 will prefer A, while a less risk averse investor (#2) will prefer B, which has higher return and higher variance.

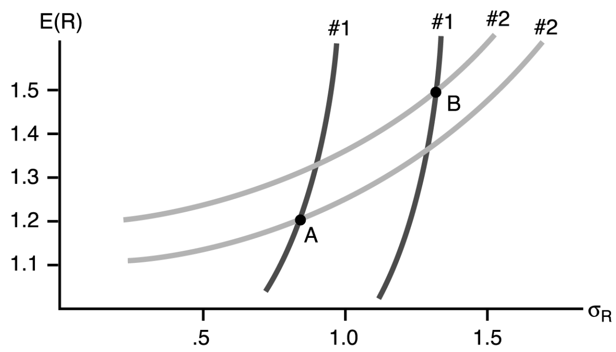


Figure S3.7 Risk-return tradeoffs

(c) The second order dominance criterion is calculated in Table S3.5

Table S3.5 (Problem 3.14) Second Order Stochastic Dominance

Return	Prob(A)	Prob(B)	F(A)	G(B)	F − G	Σ (F − G)
−.50	0	.1	0	.1	−.1	−.1
−.25	0	.2	0	.3	−.3	−.4
0	.1	0	.1	.3	−.2	−.6
.25	0	0	.1	.3	−.2	−.8
.50	.2	0	.3	.3	0	−.8
.75	0	0	.3	.3	0	−.8
1.00	.4	0	.7	.3	.4	−.4
1.25	0	0	.7	.3	.4	0
1.50	0	.4	.7	.7	0	0
1.75	0	0	.7	.7	0	0
2.00	.2	0	.9	.7	.2	.2
2.25	0	0	.9	.7	.2	.4
2.50	0	0	.9	.7	.2	.6
2.75	0	0	.9	.7	.2	.8
3.00	.1	.2	1.0	.9	.1	.9
3.25	0	0	1.0	.9	.1	1.0
3.50	0	0	1.0	.9	.1	1.1
3.75	0	0	1.0	.9	.1	1.2
4.00	0	.1	1.0	1.0	0	1.2
	1.0	1.0				

Because Σ (F − G) is not always the same sign for every return, there is no second order stochastic dominance in this case.